



B.Sc./6th Sem (H)/MTM/25(CBCS)

2025

6th Semester Examination
MATHEMATICS (Honours)

Paper : C 13-T

[Metric Spaces and Complex Analysis]

[CBCS]

Full Marks : 60

Time : Three Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

Group - A

Answer any *ten* questions : $2 \times 10 = 20$

1. Let G be an open dense subset of $\mathbb{R}$. Show that for each $x \in \mathbb{R}$, there exists $a, b \in G$ such that $x = a - b$.
2. Give an example of a sequence $\{x_n\}$ of real numbers such that $|x_n - x_{n+1}| \rightarrow 0$ (as $n \rightarrow \infty$) but $\{x_n\}$ is not Cauchy.
3. If $\{x_n\}$ is a Cauchy sequence in a metric space (X, d) , prove that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

P.T.O.

(2)

4. Show that for any subset A of a metric space (X, d) , the function $f : X \rightarrow \mathbb{R}$ given by $f(x) = d(x, A)$, $x \in X$, is uniformly continuous.

5. Show that $\lim_{z \rightarrow z_0} f(z)g(z) = 0$ if $\lim_{z \rightarrow z_0} f(z) = 0$ and if there exists a positive integer M such that $|g(z)| \leq M$ for all z in some neighbourhood of z_0 .

6. Show that the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases} \quad \text{is not}$$

continuous at $(0, 0)$.

7. Find the value of the integral $\int_0^{1+i} (x - ix^2) dz$ along the straight line from $z = 0$ to $z = 1 + i$.

8. Prove that $\int_C \frac{1}{z-a} dz = 2\pi i$.

9. Determine whether the set

$S = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1, x^2 + y^2 = 4\}$ is compact in $\mathbb{R}^2$.

(3)

10. Show that in a metric space (X, d) , the union of finite number of compact sets is compact.

11. Show that in a metric space (X, d) , no finite subset consisting more than one element is connected.

12. Show that $f(z) = \bar{z}$ is nowhere analytic.

13. Find the radius of convergence of the power series

$$\sum \frac{z^n}{n^n}.$$

14. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (3+4i)^n z^n.$$

15. Represent a function $f(z) = \frac{z}{(z-1)(z-3)}$ by a series

of positive and negative powers of $(z-1)$ which converges to $f(z)$ when $0 < |z-1| < 2$.

Group - B

Answer any **four** questions : 5×4=20

16. Show that the metric space $(P[0, 1], d)$, where

$P[0, 1]$ is the set of all polynomials over $[0, 1]$ and

$$d(f, g) = \sum_{x \in [0, 1]} |f(x) - g(x)|$$

is an incomplete metric space.

P.T.O.

17. Show that a sequence $\{x_n\}$ in a metric space (X, d) is convergent if the three sub-sequences $\{x_{2n} : n \in \mathbb{N}\}$, $\{x_{2n+1} : n \in \mathbb{N}\}$ and $\{x_{3n} : n \in \mathbb{N}\}$ of it are convergent.

18. Let f be analytic function in a region G . Show that $Re(f)$ satisfies the following equation :

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0.$$

19. If $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d) and $f: X \rightarrow X$ is a continuous map, then show that $\{(x_n, f(x_n))\}$ is a Cauchy sequence in the product metric space $X \times X$. Hence conclude that for any Cauchy sequence $\{x_n\}$ in $\mathbb{R}$, the sequence $\{(x_n, \cos x_n)\}$ is a Cauchy sequence in $\mathbb{R}^2$.

20. Evaluate

(i) $\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z^2 + 1} dz$, $t > 0$ where C is the circle $|z| = 3$ described in the positive sense.

(ii) $\int_C \frac{dz}{z(z + \pi i)}$, where C is the circle $|z + 3i| = 11$ described in the positive sense. 2+3

21. Find the domain of convergence of the following series

$$\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{n!} \left(\frac{1-z}{z} \right)^n.$$

Group - C

Answer any **two** questions : $10 \times 2 = 20$

22. (i) Let (X, d) be a metric space and A be a compact subset of X . Show that A is totally bounded.

(ii) A subset A of a metric space (X, d) is totally bounded if and only if every sequence in A has a Cauchy subsequence.

(iii) If $\mathcal{F}$ be a family of compact sets with finite intersection property in a metric space (X, d) , then show that $\bigcap \mathcal{F} \neq \emptyset$. 2+5+3

23. (i) Show that the map $f : [0, 1] \rightarrow [0, 1]$, defined by

$f(x) = x - \frac{x^2}{2}$, $x \in [0, 1]$ is not a contraction map.

(ii) Let D be a non-empty open subset of the Euclidean plane $\mathbb{R}^2$, $f : D \rightarrow \mathbb{R}$ be a continuous map which satisfies a Lipschitz condition with respect to the second variable, i.e.,

$$|f(x, y_1) - f(x, y_2)| \leq k |y_1 - y_2| \quad \text{for all}$$

$(x, y_1), (x, y_2) \in D$ and for some $k > 0$, and let

$$(x_0, y_0) \in D.$$

Show that the differential equation $\frac{dy}{dx} = f(x, y)$

P.T.O.

has a unique solution $y = g(x)$ in the interval $[x_0 - c, x_0 + c]$, for a suitable $c > 0$, such that the boundary condition $g(x_0) = y_0$ is satisfied. 2+8

24. (i) Suppose $f(z)$ is analytic in a domain Ω and $C = \{z : |z - a| = R\}$ is contained in Ω . Then

prove that $|f^n(a)| \leq \frac{n! M_R}{R^n}$, $n = 0, 1, 2, \dots$ where

$$M_R = \max_{z \in C} |f(z)|.$$

- (ii) Show that every bounded entire function is constant.

- (iii) Let $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$, $a_n \neq 0$.

Show that there exists a point z_0 in C such that

$$p(z_0) = 0. \quad 3+3+4$$

25. (i) Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

- (ii) Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right) \text{ in the domain } 0 < |z| < \infty.$$

5+5