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B.Sc./3rd Sem/MTM/25(NEP)

2025

3rd Semester Examination (CCFUP : NEP)

MATHEMATICS

Paper : MI 3-T (Minor)

Algebra (For Single Core) / Differential Equations and Vector Calculus (For Multidisciplinary Studies Major)

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Algebra]

(For Single Core Major)

Group - A

Answer any *ten* questions : $2\times10=20$

1. Find the value of $\phi(2021)$, where ϕ is Euler's Phi function.

2. Apply Descarte's rule of signs to determine the number of complex root of $x^4+2x^2+3x=1$.

3. Obtain polar form of the complex number $z=-1+i$. What is the principal argument of z?

P.T.O.

4. If n be a positive integer greater than 1, prove that $n(n+1)^2 > 4(n!)^{3/n}$.

5. Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 7 & 1 \\ 3 & 1 & 2 & -1 \\ 1 & 6 & 2 & 1 \end{pmatrix}$.

6. State the fundamental theorem of arithmetic. Write down the prime factorization of 2025.

7. If a, b and c are positive real numbers, prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$.

8. Express the gcd (91, 52) as $91x + 52y$ for some $x, y \in \mathbb{Z}$.

9. Show that a non-singular matrix cannot have zero eigen value.

10. Define linear map on a vector space. Check whether $T:\mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = xy$, for all $(x, y) \in \mathbb{R}^2$, is linear or not.

11. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the value of $\sum \alpha^5$.

12. Let $S = \{\alpha, \beta, \gamma\}$ and $T = \{\alpha, \beta, \alpha+\beta, \beta+\gamma\}$ be subsets of a real vector space V. Show that $L(S) = L(T)$.

13. Prove that $(1+2+\cdots+n)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)>n^2,\ n>1$.

14. If one of the roots of the equation $2x^3-x^2-18x+9=0$ is $\frac{1}{2}$ and the other two roots are equal in magnitude but opposite in sign, find these roots.

15. Let V be a vector space of all real valued functions over the field $\mathbb{R}$. Verify whether $W=\{f\in V: f(3)=1+f(4)\}$ is a subspace of V.

Group - B

Answer any ***four*** questions : 5×4=20

16. Show that $S=\left\{X=\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \,\middle|\, AX=0\right\}$, where $A=\begin{bmatrix}1&1&1\\1&2&3\\2&1&0\end{bmatrix}$ is a subspace of $\mathbb{R}^3$. Find a basis for S.

17. If z is a variable complex number such that the amplitude of $\frac{z-i}{z+1}$ is $\frac{\pi}{4}$, then show that z lies on a circle in the complex plane.

P.T.O.

18. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$ and let W be the set of all linear combinations of the columns of A.

(a) Is b in W? Why or why not?

(b) Show that the second column of A is in W. 4+1

19. For what values of k the planes $x + y + z = 2$, $3x + y - 2z = k$ and $2x + 4y + 7z = k + 1$ form a triangular prism?

20. If $a_1, a_2, \ldots\ldots, a_n$ be all positive real numbers and $s = a_1 + a_2 + \cdots\cdots + a_n$, prove that

$$\left(\frac{s-a_1}{n-1}\right)\left(\frac{s-a_2}{n-1}\right)\cdots\cdots\cdots\left(\frac{s-a_n}{n-1}\right) = a_1.a_2\ldots\ldots.a_n,\ \text{unless}$$

$a_1 = a_2 = \ldots\ldots = a_n$.

21. Find the values of k for which the system of equations

$$\begin{aligned} x+y-z&=1 \\ 2x+3y+kz&=3 \\ x+ky+3z&=2 \end{aligned}$$

has (i) no solution, (ii) more than one solutions, (iii) unique solution.

Group - C

Answer any *two* questions : 10×2=20

22. (a) Let $A=\begin{bmatrix}-8 & -2 & -9\\ 6 & 4 & 8\\ 4 & 0 & 4\end{bmatrix}$ and $W=\begin{bmatrix}2\\ 1\\ -2\end{bmatrix}$. Determine if W is in the column space of A. Is W in the null space of A? Justify.

(b) Let α, β and γ be the roots of the equation $x^3+px^2+qx+r=0$. Then find (a) the value of $\sum\alpha^2\beta$, (b) an equation whose roots are $\alpha\beta+\beta\gamma$, $\beta\gamma+\gamma\alpha$ and $\gamma\alpha+\alpha\beta$. (3+2)+(3+2)

23. (a) For what values of k the planes $x-4y+5z=k$, $x-y+2z=3$ and $2x+y+z=0$ intersect in a line? Find the equation of the intersecting line in that case.

(b) The matrix of a linear mapping $T:\mathbb{R}^3\rightarrow\mathbb{R}^3$ relative to the ordered basis $\{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ of $\mathbb{R}^3$ is $\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$. Find the matrix of T relative to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$. (4+1)+5

P.T.O.

24. (a) Let $f : A \to B$ and $g : B \to C$ be two mappings. Prove that (a) if $g \circ f$ is injective and f is surjective then g is injective, (b) if $g \circ f$ is surjective and g is injective then f is surjective.

(b) Solve the equation by Ferrari's method :

$$x^4 + 12x^3 + 54x^2 + 96x + 40 = 0.$$

(c) Find the values of $\log(-i)^{1/2}$. (2+2)+4+2

25. (a) Reduce the matrix $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ to a row-reduced Echelon form and hence find its rank.

(b) Find the eigen values and the corresponding eigen vectors of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(c) If X and Y are two non-empty sets and $f : X \to Y$ is a mapping, then for any subsets A and B of Y show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. 4+4+2

Paper : MI 3-T

Differential Equations and Vector Calculus

(For Multidisciplinary Studies Major)

Group - A

Answer any *ten* questions : 2×10=20

1. Determine whether $x=0$ or $x=1$ is an ordinary point of the differential equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$ for any positive integer n.

2. Find an equation for the plane determined by the points $P_1(2,-1,1)$, $P_2(3,2,-1)$, and $P_3(-1,3,2)$.

3. Find the general solution of

$$\frac{dx}{cy-bz}=\frac{dy}{az-cx}=\frac{dz}{bx-ay}$$

4. Prove that

$F=\left(y^2\cos x+z^3\right)i+(2y\sin x-4)j+\left(3xz^2+2\right)k$ is a conservative force field.

5. What can one say about the general solution of $y''+16y=0$ if two particular solutions are known to be $y_1=\sin 4x$ and $y_2=\cos 4x$?

P.T.O.

6. For what values of a and b, all real solutions of the differential equation $\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=\cos x$ are periodic.

7. Find the volume of the parallelepiped whose edges are represented by :

$$\boldsymbol{A}=2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k},\ \boldsymbol{B}=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k},\ \boldsymbol{C}=3\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}.$$

8. Find the general solution of the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+9y=0.$

9. Show that $\nabla r^n = nr^{n-2}\boldsymbol{r}$ (symbols have their usual meaning).

10. Show that the Wronskian of the function $f_1(x)=x^2$ and $f_2(x)=x|x|$ is zero.

11. Determine whether the set $\{1-x,\ 1+x,\ 1-3x\}$ is linearly dependent on $(-\infty, \infty)$.

12. Find an equation of the tangent plane to the surface $x^2+2xy^2-3z^3=6$ at the point $P(1, 2, 1)$.

13. Determine the singular points of the following differential equation

$$x^2(x-1)^2\frac{d^2y}{dx^2}+2(x-2)\frac{dy}{dx}+(x+3)y=0.$$

14. Show that $f(t, x) = (3t + 2x_1, x_1 - x_2)$ on $S : \{|t| < \infty, \|x\| < \infty\}$ satisfying Lipschitz condition.

15. Find the constant a so that the following vectors are coplanar :

$$3i - 3j - k, \; -3i - 2j + 2k, \; 6i + aj - 3k.$$

Group - B

Answer any ***four*** questions : 5×4=20

16. Apply the method of variation of parameters to solve :

$$\frac{dy}{dx} + y = \sec^3 x \tan x$$

17. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = -t - 5$ where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction of $\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

18. Find a recurrence formula for the power series solution around $t = 0$ for the differential equation

$$\frac{d^2y}{dt^2} + (t-1)\frac{dy}{dt} + (2t-3)y = 0.$$

19. Prove that $\nabla \times (\nabla \times A) = -\nabla^2 A + \nabla(\nabla . A)$, (symbols have their usual meaning).

P.T.O.

20. Solve the following differential equation by the undetermined coefficients :

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}.$$

21. Given $A = (yz + 2x)i + xzj + (xy + 2z)k$. Evaluate $\int_C A.dr$ along the curve $x^2 + y^2 = 1$, $z = 1$ in the positive direction from (0, 1, 1) to (1, 0, 1).

Group - C

Answer any *two* questions : 10×2=20

22. (a) Solve $\left(D^2 - 4D + 4\right)y = 8x^2 e^{2x} \sin 2x$.

(b) Prove that there exists a set of n linearly independent solutions of $\dot{x}(t) = A(t)x(t)$.

(c) Find the solution for the system

$$\frac{dy_1}{dt} = -3y_1 + 2y_2, \quad \frac{dy_2}{dt} = -2y_1 + 2y_2.$$ 4+2+4

23. (a) Let F be a conservative force field such that $F = -\nabla\phi$. Suppose a particle of constant mass m is to move in this field. If A and B are any two points in space, prove that

$$\phi(A) + \frac{1}{2}mv_A^2 = \phi(B) + \frac{1}{2}mv_B^2$$

where v_A and v_B are the magnitudes of the velocities of the particle at A and B, respectively.

(b) The equation of motion of a particle are given by $\frac{dx}{dt}+\omega y=0$ and $\frac{dy}{dt}-\omega x=0$. Find the path of the particle and show that it is a circle. 4+6

24. (a) Use the method of Fröbenius to find one solution near $x=0$ of

$$x^2y''-xy'+y=0.$$

(b) Suppose a force field is given by

$$\boldsymbol{F}=(2x-y+z)\boldsymbol{i}+\left(x+y-z^2\right)\boldsymbol{j}+(3x-2y+4z)\boldsymbol{k}$$

Find the work done in moving a particle once around a circle $\boldsymbol{C}$ in the xy-plane with its center at the origin and a radius of 3. 5+5

25. (a) Evaluate $\iint_S \boldsymbol{A}.\boldsymbol{n}\, dS$, where $\boldsymbol{A}=18z\boldsymbol{i}-12\boldsymbol{j}+3y\boldsymbol{k}$ and S is that part of the plane $2x+3y+6z+12$, which is located in the first octant.

(b) Solve the following system of differential equation :

$$\frac{dx}{dt}+4x+3y=t,\ \frac{dy}{dt}+2x+5y=e^t. \qquad 4+6$$
