Total Pages : 20 B.Sc./5th Sem (G)/MATH/23(CBCS)

2023

5th Semester Examination

MATHEMATICS (General)

Paper : DSE 1A/2A/3A-T

(CBCS)

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

(Complex Analysis)

Group - A

(4 Marks)

1. Answer any *ten* of the following questions : $2\times10=20$

(a) If $z \neq (0,0)$ be any complex number, then prove that $\left|\frac{1}{z}\right| = \frac{1}{|z|}$.

(b) For any complex number $z = (x, y)$, prove that $|x|+|y| \leq \sqrt{2}|z|$.

P.T.O.

(c) Evaluate by the method of residue :

$$\int_C \frac{dz}{(z^2+1)(z-4)}$$

where $C: |z| = 3$.

(d) Determine the orthocentre of the triangle with vertices z_1, z_2 and z_3.

(e) Show that the function $f(z) = \frac{z - \sin z}{z^3}$ has a removal singularly at $z = 0$.

(f) Show that the function $f(z) = z^3$ is analytic in a domain of the complex plane C.

(g) Determine the radius of convergence of the power series —

$$\sum \frac{z^n}{n!}$$

(h) Show that the function $f(z) = z - 1$ has no fixed point in $\mathbb{C}$.

(i) Find residue of $\phi(z) = \cot z$ at the point $z_n = n\pi$ for $n = 1, 2, \ldots$

(j) If the complex number $\frac{z-i}{z+i}$ is purely imaginary, then show that the point z lies on a circle with centre at the origin and radius 1.

(k) Give an example of a continuous function of complex variable which is not analytic.

(l) Using Cauchy's integral formula, evaluate $\int_{|z|=1} \frac{\cos(2\pi z)}{(2z-1)(z-2)} dz.$

(m) If $a = \cos\theta + i\sin\theta$, obtain the value of θ in $[0, \pi]$ such that $a^3 = i$.

(n) Evaluate $\int_c \log z dz$ where $c : |z| = 1$.

(o) State Liouville's theorem.

Group - B

2. Answer any *four* of the following questions : 5×4=20

(a) Find the residue of $F(z) = \frac{\cot z . \cot h z}{z^3}$ at $z = 0$.

(b) If the real part of the complex number $\frac{z-i}{z-1}$ is zero, then show that the complex number z lies on the circle with centre $\frac{1+i}{2}$ and radius $\frac{1}{\sqrt{2}}$.

P.T.O.

(c) Consider the function f defined by —

$$f(z)=\begin{cases} 0 & z=0 \\ \dfrac{x^3-y^3}{x^2+y^2}+i\dfrac{x^3+y^3}{x^2+y^2}, & z\neq 0. \end{cases}$$

Show that the function f satisfies the Cauchy-Riemann equations at the origin, but is not differential at $z = 0$.

(d) State and prove the fundamental theorem of integral calculus in the complex plane.

(e) Show that radius of convergence of the series —

$\frac{1}{2}z+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+...$ is $\frac{3}{2}$.

(f) State and prove the Laurent's theorem.

Group - C

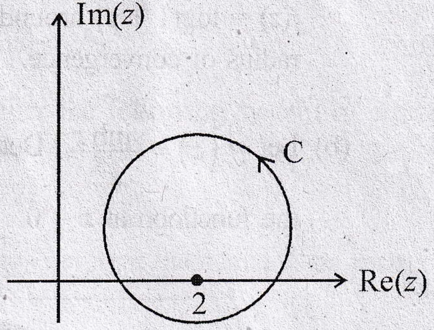
3. Answer any *two* of the following questions : 10×2=20

(i) (a) State and prove the Cauchy-Goursat theorem.

(b) Let $u(x,y)=e^x\cos y$. Determine a function $v(x, y)$ such that the function $f = u + iv$ is analytic. 5+5=10

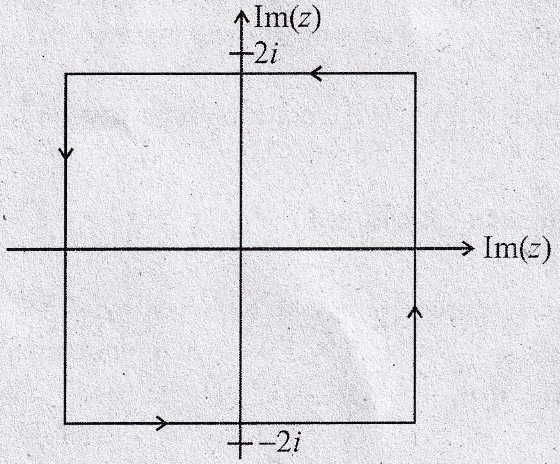
(ii) (a) Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent series, which is valid in $|z| < 1$.

(b) Compute $\int \frac{e^{z^2}}{z-2} dz$, over the curve C shown below.



5+5

(iii) (a) Compute $\int \frac{\cos z}{z\left(z^2+8\right)} dz$ over the outer contour shown below :



P.T.O.

(b) Compute $\int z^2 dz$ along the straight line from 0 to $1 + i$. 5+5=10

(iv) (a) Find the Taylor series for the function $f(z) = \log(1 + z)$ around $z = 0$. Give also the radius of convergence.

(b) Let $f(z) = \frac{\sinh z}{z^5}$. Determine the residue of the functioon at z = 0. 5+5

OR

(Matrics)

Group - A

1. Answer any *ten* questions : 2×10=20

(a) Check whether the following system of equations is consistent or not : $3x_1 + 2x_2 + 3x_3 = 5$, $2x_1 + x_2 + 2x_3 = 2$, $x_1 + x_2 + x_3 = 1$.

(b) Verify Cayley-Hamilton theorem for the matrix :

$$\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}.$$

(c) Find inverse of the matrix $M = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ by using Cayley-Hamilton Theorem.

(d) Prove that an elementary row operation of the first kind does not alter the row rank of a matrix.

(e) If a linear transformation $T : R^3 \to R^3$ is defined as

$T(x, y, z) = \left(\frac{x}{3}, \frac{y}{4}, 0\right)$, then find the rank of T.

(f) For what value of p the following system of equations $x + y + z = 2$, $x + 3y + 2z = 5$, $2x + y + 3z = 1$, $3x - 2y + z = p$ (p : real) is solvable.

P.T.O.

(g) Prove that the eigenvalues of a real symmetric matrix are all real.

(h) Find the eigenvalues of the following matrix :

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

(i) Find the invariant points of the transformations defined by $x' = 1 + y, y' = 4x + 10$.

(j) Determine whether the set {[1, 1, 3], [2, –1, 3], [0, 1, 1], [4, 4, 3]} is linearly independent.

(k) Prove that a matrix and its transpose have the same eigenvalues.

(l) Show that $W = \{(x, y, z) \in \mathbb{R}^3 : x - 3y + 4z = 0\}$ is a subspace of R^3.

(m) What is the rank of the following matrix :

$$\begin{bmatrix} 1 & -3 & -2 & 3 \\ -2 & -1 & 4 & 1 \\ 3 & 5 & 1 & 2 \\ 1 & 4 & 5 & 3 \end{bmatrix}.$$

(n) Express (5, 2, 1) as a linear combination of (1, 4, 0), (2, 2, 1) and (3, 0, 1).

(o) If S be a real skew-symmetric matrix of order n. prove that $(I_n + S)$ is non-singular.

2. Answer any *four* questions : 5×4=20

(a) Prove that the row rank and column rank of any matrix are identical.

(b) Show that the set of vectors {(1, 2, 2), (1, −1, 2), (1, 0, 1)} forms a basis in R^3.

(c) Find the inverse of the matrix $A = \begin{bmatrix} 5 & 1 & -6 \\ 0 & 4 & 2 \\ 1 & 3 & -4 \end{bmatrix}$ using elementary row operations.

(d) Find all real values of λ for which the rank of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{bmatrix}$ is 2.

(e) If $\{\alpha, \beta, \gamma\}$ be the basis of a vector space R^3, then show that $\{\alpha, \beta, \beta + \gamma, \gamma + \alpha\}$ is also a basis set for R^3.

(f) Find the rotation matrix which rotates a vector $(x, y) \in \mathbb{R}^2$ through an angle of θ in the counter

P.T.O.

clockwise direction. Is $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ a rotation matrix? Justify.

3. Answer any *two* questions : 10×2=20

(a) (i) Define the eigenbasis for a square matrix.

(ii) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$, check whether it has eigenbasis or not. 2+8

(b) (i) Let $T : R^2 \rightarrow R^3$ be a linear transformation, such that the matrix representation of T is $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ relative to the basis {(1, 0, 0), (1, 1, 0), (1, 1, 1)} of R^3 (domain) and {(0, 1, 1), (1, 0, 1), (1, 1, 0)} of R^3 (co-domain). Then find T. Also find $T(1, 2, 3)$.

(ii) Let $T : R^2 \rightarrow R^3$ be a linear transformation, which transforms the order basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)} to {(3, 2, 1), (1, 3, 2), (2, 3, 7)}. Then find the matrix of T. 8+2

(c) (i) Show that the following matrix is diagonalisable

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also, write the diagonal matrix. Find, in the form $y = mx + c$, the equations of all invariant lines of the transformation given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 3+2+5

(d) (i) State the condition for which a non-homogeneous system of linear equations is consistent.

(ii) If $\lambda \neq 14$, then show that the system of equations

$5x + 2y - z = 1$, $2x + 3y + 4z = 7$, $4x - 5y + \lambda z = \lambda - 5$ has unique solution $(0, 1, 1)$. 2+8

P.T.O.

OR

(Linear Algebra)

Group - A

1. Answer any *ten* questions : 2×10=20

(a) Examine if the set of vectors {(2, 1, 1), (1, 2, 2), (1, 1, 1)} is linearly dependent in R^3.

(b) Find the rank of the matrix $\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 6 \\ 3 & 0 & 7 & 9 \end{pmatrix}$.

(c) For a vector space V over the field F prove that $(-c)\,v = c(-v) = -(cv)$ where $c \in F, v \in V$.

(d) Define improper subspace and trivial subspace for a vector space $V(F)$.

(e) Prove that intersection of two subspace of a vector space $V(F)$ is a subspace of $V(F)$.

(f) Define linear sum and direct sum for a vector space $V(F)$.

(g) Prove that every subset of a linearly independent set is linearly independent.

(h) Find a basis of the subspace

$$W = \{(x, y, z) \mid x + y - z = 0,\ 2x - y - z = 0\}$$

(i) Define identify mapping and zero mapping on a vector space $V(F)$.

(j) If $V(F)$ and $W(F)$ are vector spaces and $T : V \to W$ is a linear mapping then show that $T(\theta) = \theta'$.

(k) A mapping $T: R^2 \to R^2$ defined by
$T(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$
where $(x,y) \in R^2, \alpha$ is constant. Examine whether T is linear or not.

(l) State isomorphism theorem.

(m) Show that the vectors (1, 0, 0) and (0, 1, 0) form a basis of the vector space $V_3 = \{(x, y, 0), x, y \in R\}$.

(n) Let $T_1 : R^2 \to R^2$ and $T_2 : R^2 \to R^2$ are two linear operators defined by $T_1(x, y) = (x + y, x - y)$ and $T_2(x, y) = (-y, -x)$. Evaluate $2T_1 - 2T_2$ and $T_1 o T_2$.

(o) Examine the nature of intersection of the planes $2x_1 + x_2 - 2x_3 = 3$; $x_1 - 2x_2 + x_3 = 3$; $2x_1 + 2x_2 - 4x_3 = 1$.

2. Answer any *four* questions : $5 \times 4 = 20$

(a) Show that the set $\{(1, i, 0), (2i, 1, 1), (0, 1 + i, 1 - i)\}$ forms the basis of V_3 over the complex field C.

(b) Let V be a vector space over a field F and W be a subspace of V. Prove that $\dim V/W = \dim V - \dim W$.

P.T.O.

(c) Define Ker T for a linear mapping. Prove that ker T of a linear mapping $T : V \to W$ is subspace of V.

(d) Find the linear operator $T : R^2 \to R^2$, if it satisfies the conditions $T(1, 0) = (1, -1)$, $T(0, 2) = (4, -2)$.

(e) Let A and B be two matrices of same order so that $A + B$ is defined. Prove that $rank\ (A + B) \leq rank\ (A) + rank\ (B)$.

(f) Let (x_1, x_2, x_3) be an ordered basis of a real vector space $V(F)$ and a linear mapping $T : V \to V$ is defined by $T(x_1) = x_1$, $T(x_2) = x_1 + x_2$, $T(x_3) = x_1 + x_2 + x_3$. Find the matrix of T^{-1}.

3. Answer any *two* questions : 10×2=20

(a) (i) If $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a linearly dependent set of a vectors of a finite dimensional vector space $V(F)$, then prove that there exist a basis B of $V(F)$, so that $B \subset S$.

(ii) For what real value of k does the set

$S = \{(k, 1, 1), (1, k, 1), (1, 1, k)\}$ form a basis of R^3?

(iii) Find the co-ordinate vector of $\alpha = (1, 3, 1)$ relative to the order basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 5+2+3

(b) Let $L(V, W)$ be a set of all linear mappings with domain V and co-domain W. Prove that $L(V, W)$ is a linear space. Also, find dim $L(V/W)$. 5+5

(c) (i) Find a orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix where

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}.$$

(ii) State and prove Schwarz's inequality in a Euclidean space V. 5+5

(d) (i) Let U and W be subspaces of a vector space $V(F)$. Prove that $U \cup W$ is a subspace iff either $U \subset W$ or $U \supset W$.

(ii) Let $S = \{\alpha, \beta, \gamma\}$ and $T = \{\alpha, \beta, \alpha + \beta, \beta + \gamma\}$ be two subspaces of a vector space V. Show that $L(S) = L(T)$.

(iii) The matrix of a linear mapping $T : R^3 \to R^2$ relative to the order bases {(0, 1, 1), (1, 0, 1), (1, 1, 0)} of R^3 and {(0, 1), (1, 1)} of R^2 is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T. Also, find the matrix of T relative to the order bases {(1, 1, 0), (1, 0, 1), (0, 1, 1)} of R^3 and {(1, 1), (0, 1)} of R^2. 3+3+4

P.T.O.

OR

(Vector Calculus and Analytical Geometry)

Group - A

1. Answer any *ten* questions : 2×10=20

(a) Two vectors $\vec{a}$ & $\vec{b}$ are such that

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2.$$

Prove that they are orthogonal.

(b) Show that the vector $\left(\vec{\alpha} - \frac{\vec{\alpha}.\vec{\beta}}{|\vec{\beta}|^2}\vec{\beta}\right)$ is perpendicular to the vector $\vec{\beta}$.

(c) Find the unit normal vector to the surface $3x^2y - y^3z^2 = 0$ at (1, –2, –1).

(d) Find the value of a for which the parabola $x^2 = 4ay$ passes through the point (1, 2) & hence find the coordinate of its focus and the length of the latus rectum.

(e) Find the equation of the ellipse one of whose foci is (–1, 1), eccentricity is $\frac{1}{2}$ & the corresponding directrix is $y = x - 3$.

(f) Find the equation of the cylinder whose generating line is parallel to the Z-axis and the guiding curve is $5x^2-2y^2+7z^2=1, 3x+2y-z=5$

(g) Determine the values of h for which the plane $x + y + z = h$ is a tangent plane to the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z = 6$.

(h) If $\vec{r}=\sin t\hat{i}-\cos t\hat{j}+t\hat{k}$ and $\vec{s}=\cos t\hat{i}+\sin t\hat{j}+t\hat{k}$ then find $\frac{d}{dt}(\vec{r}.\vec{s})$ and $\frac{d}{dt}(\vec{r}\times\vec{s})$.

(i) Show that the vector function $\vec{F}(x,y,z)=3y^4z^2\hat{i}+4x^3z^2\hat{j}-3x^2y^2\hat{k}$ is solenoidal.

(j) Check whether the vector function $\vec{A}(x,y,z)=\sin y\hat{i}-\sin x\hat{j}+e^z\hat{k}$ is irrotational or not.

(k) Find the curl $\vec{A}$ of the vector point function $\vec{A}=x^2z\hat{i}-2y^3z\hat{j}+xy^2z\hat{k}$ at (1, 1, 1).

(l) Find the maximum value of the directional derivative of $\varphi=x^2+z^2-y^2$ at the point (1, 3, 2). Find also the direction in which it occurs.

(m) If the vectors $\vec{A}$ and $\vec{B}$ be irrotational, then show that the vector $\vec{A}\times\vec{B}$ is solenoidal.

P.T.O.

(n) Show that the equation

$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$

represents an ellipse whose centre is at (2, 3).

(o) Find the foci, directrices, eccentricity and the length of latus rectum of the ellipse $9x^2 + 25y^2 = 225$.

Group - B

2. Answer any *four* of the following questions : 5×4=20

(a) Show that the vector $r^n\vec{r}$ is irrotational where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ & $|\vec{r}| = r$.

(b) Show that the plane $y - 6 = 0$ intersects the hyperbolic paraboloid $\frac{x^2}{5} - \frac{y^2}{4} = 6z$ in a parabola.

(c) (i) If $\frac{d\vec{a}}{dt} = \vec{r} \times \vec{a}$ and $\frac{d\vec{b}}{dt} = \vec{r} \times \vec{b}$, then show that $\frac{d}{dt}\left(\vec{a} \times \vec{b}\right) = \vec{r} \times \left(\vec{a} \times \vec{b}\right)$, where $\vec{r}$ is a constant vector and $\vec{a}, \vec{b}$ are vector function of a scalar variable t.

(ii) If $\vec{\alpha} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{\beta} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ have their usual meaning, then find $\frac{d}{dt}\left(\vec{\alpha} \times \frac{d\vec{\beta}}{dt}\right)$.

2+3

(d) Reduce the following equation to its canonical forms and determine the nature of the conic

$$4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$$

(e) (i) If $\vec{f} = (2x^2y - x^4)\hat{i} + (e^{xy} - x\sin y)\hat{j} + (y^2\cos x)\hat{k}$, then verify that $\vec{f}_{xy} = \vec{f}_{yx}$.

(ii) Show that the vector $\dfrac{\vec{r}}{r^3}$ is both solenoidal and irrotational, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$.

2+3

(f) Show that the length of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ intercepted on the straight lines $y = mx + c$ is $\dfrac{2ab\sqrt{(1+m^2)(a^2m^2+b^2-c^2)}}{a^2m^2+b^2}$

Group - C

3. Answer any *two* of the following questions : 10×2=20

(a) (i) Show by vector method that the join of the middle points of two sides of a triangle is parallel to the third side and is half of its length.

P.T.O.

(ii) In any triangle ΔABC, with usual notations prove by vector method that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$ 5+5

(b) (i) Show by vector method that if two medians of a triangle be equal then the triangle is isosceles.

(ii) Position vectors of P & Q referred to the origin are $\left(\vec{i} - 2\vec{j} + \vec{k}\right)$ and $\left(3\vec{i} - 5\vec{j} + 2\vec{k}\right)$ respectively. Find the scalar area of ΔOPQ. 5+5

(c) (i) Show that the vector

$$\vec{F} = \left(6xy + z^3\right)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

is irrotational. Find the scalar function φ for this field, such that $\vec{F} = \vec{\nabla}\varphi$.

(ii) If $\vec{F} = \left(3x^2y - z\right)\hat{i} + \left(xz^3 + y^4\right)\hat{j} - 2x^3z^2\hat{k}$, then evaluate *grad div* $\vec{F}$ at the point (2, –1, 0). 5+5

(d) Prove that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two parallel straight lines, if $h^2 = ab$ and $bg^2 = af^2$.

Also show that the distance between them is

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}}.$$ 5+5