Total Pages : 10

B.Sc./5th Sem (G)/MATH/23(CBCS)

2023

5th Semester Examination

MATHEMATICS (General)

Paper : SEC 3-T

[CBCS]

Full Marks : 40 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Number Theory]

1. Answer any ***five*** questions : $2\times5=10$

(a) Find $\tau(180)$ and $\sigma(180)$.

(b) Prove that for each positive integer $n \geq 1$,

$$\Sigma_{d|n} \mu(d) = \left[\frac{1}{n}\right].$$

(c) State Möbius inversion formula.

(d) Verify that 50! terminates in 12 zeros.

(e) If gcd $(a, b) = 1$ then prove that gcd $(a^2, b) = 1$.

(f) Compute $\varphi(1080)$.

P.T.O.

(g) Use division algorithm to prove that the square of an odd integer is of the form $8k+1$ where $k \in \mathbb{Z}$.

(h) Use Euler's theorem to find the unit digit in 3^{100}.

2. Answer any ***four*** questions : 5×4=20

(a) Prove that $m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 (\bmod mn)$ where $\gcd(m, n) = 1$ and $m, n \in \mathbb{Z}$.

(b) For a positive integer r, prove that the product of any r consecutive positive integers is divisible by r.

(c) For a positive integer $n \geq 3$, establish the formula

$\Sigma_{k=1}^{n} \mu(k!) = 1$, where μ is the Möbius μ function.

(d) Use the theory of congruence to prove that for any integer $n \geq 1$, $43 \mid 6^{n+2} + 7^{2n+1}$.

(e) Establish that there are infinitely many primes of the form $4k-1$, $k \in \mathbb{Z}$.

(f) Find the general solution in integers and the least positive integral solutions of the equation

$$63x - 55y = -1 \text{ where } x, y \in \mathbb{Z}.$$

3. Answer any ***one*** question : 10×1=10

(a) State and prove Chinese Remainder Theorem.

(b) State and prove Fundamental Theorem of Arithmetic.

OR

[Bio-Mathematics]

1. Answer any ***five*** questions : 2×5=10

(a) Discuss the Malthus population growth model for Single Species.

(b) What do you mean by functional response and numerical response of a prey-predator system?

(c) Write down Holling type-I and type-II response function with their graphical representation.

(d) Define discrete and continuous models.

(e) Find the steady states of the following system of equations and determine the Jacobian of the system for these steady states :

$$\frac{dx}{dt} = x - xy,$$
$$\frac{dy}{dt} = xy - y.$$

(f) Find the equilibrium points of the difference equation $x_{t+1} = \frac{\alpha x_t^2}{x_t^2 + a}(\alpha > 0, a > 0)$ provided $\alpha < 2\sqrt{a}$.

P.T.O.

(g) Suppose the characteristic equation of a discrete system corresponding to an equilibrium point is $p(\lambda) = \lambda^3 - 1.3\lambda^2 - 0.08\lambda + 0.24$. Verify whether the system is stable.

(h) Write down the Routh-Hurwitz criterion of stability for a three dimensional system.

2. Answer any ***four*** questions : 5×4=20

(a) For the model $\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{K}\right) - Ex$ with $x(0) = K$ where α, E and K are constants. Determine $x(t)$ explicity. Show that $x > K\left(1 - \frac{E}{\alpha}\right)$. If $E \leq \alpha$ then $x \to K\left(1 - \frac{E}{\alpha}\right)$ as $t \to \infty$ whereas if $E > \alpha$, then $x \to 0$ as $t \to \infty$.

(b) Consider a logistically growing population. If the population is harvested at a rate proportional to the population at that instant, formulate the modified logistic equation taking into account the effect of harvesting. If the initial population is K, obtain the solution and discuss it as $t \to \infty$.

(c) Consider the difference equation $x_{t+1} = \frac{ax_t}{b + x_t}$, a, $b > 0$. Find the fixed points of this system and discuss about their stability.

(d) Let $\overline{x}$ be an equilibrium point of the first order difference equation $x_{t+1} = f(x_t)$ where f is sufficiently smooth. Find the restriction for which the equilibrium will be asymptotically stable and unstable.

(e) What is diffusion in mathematical model? Give an example of one species model with diffusion. 3+2

(f) Define a SIR model with generalized assumptions and hence analysize the stability of its equilibrium points. 3+2

3. Answer any ***one*** question : 10×1=10

(a) (i) Suppose the population x and y satisfy the equation :

$$\dot{x} = x(60 - 4x - 3y)$$
$$\dot{y} = y(42 - 2y - 3x)$$

Find all the critical points of the system. Which critical point represents the possibility of co-existence of two species? Discuss the type of stability of that critical point. 8

(ii) In the epidemic model, if the contact rate is 0.002 and the number of susceptibles be 4000 initially, determine the number of susceptibles left after 2 weeks. 2

P.T.O.

(b) Consider the following system

$$\frac{dx}{dt} = x\left(1 - \frac{x}{k}\right) - axy$$

$$\frac{dy}{dt} = exy - py$$

where k, a, e and p are all positive constants.

(i) Find corresponding steady states and Jacobean matrix around any fixed point.

(ii) Discuss the stability of interior steady state only. (3+2)+5

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OR

[Mathematical Modelling]

1. Answer any ***five*** questions : 2×5=10

(a) A car on the free way accelerates according to $15\cos(\pi t)$ where t is measured in hours. Determine the velocity of the car at any time t, if it has an initial speed 51 km/hour.

(b) Write down Holling type-I and type-II response function with their graphical representation.

(c) Find the steady states of the following system

$$\dot{x} = e^{1-x} - 1$$

$$\dot{y} = (2-y)e^{x}$$

(d) Write down the differential equation of mass on a spring under undamped free vibration.

(e) A 12 volt battery is connected to a simple series circuit in which the inductance is 0.5 H and resistance is $10\,\Omega$. Determine the current i.

(f) Show that the two solutions $x = e^{t}$ and $x = e^{-t}$ of $x'' - x = 0$ are linearly independent for all values of t.

(g) Given the mass spring system represented by the equation $y'' + 4y' + ky = 0$. For what value of K, the system is critically damped?

P.T.O.

(h) If the population of a country doubles in 50 years, in how many years will it treble? Given that the rate of increase is proportional to the number of inhabitants.

2. Answer any ***four*** questions : $5 \times 4 = 20$

(a) Assume that a spherical raindrop evaporates at a rate proportional to its surface area. If its radius is 3 mm and one hour later has been reduced to 2 mm. Find an expression for the radius of the raindrop at any time t.

(b) Initially 50 pounds of salt is dissolved in a large tank having 300 gallon of water. A brine solution is pumped into the tank at a rate of 3 gal/m and well stirred solution is then pumped out at the same rate. If the concentration of the solution entering is 2 *lb*/gal, find the amount of salt in the tank at any time.

(c) A 64-*lb* weight is placed upon the lower end of a coil spring suspended from a rigid beam. The weight comes to rest in its equilibrium position, thereby stretching the spring 2 ft. The weight is then pulled down 1 ft below its equilibrium position and released from rest at $t = 0$.

(i) What is the position of the weight at $t = \frac{5\pi}{12}$? How and which way is it moving at the time?

(ii) At what time is the weight 6 inches above its equilibrium position and moving downwards? What is the velocity at such time?

(d) Consider a logistically growing population. If the population is harvested at a rate proportional to the population at that instant, formulate the modified logistic equation taking into account the effect of harvesting. If the initial population is K, obtain the solution and discuss it as $t \to \infty$.

(e) Derive the differential equation of electric circuit problem in the form

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E$$

where E = voltage, q = charge, R = resistance, L = inductance, C = capacitance.

(f) A circuit has in series an electromotive force $E = 100 \sin tv$, a resistor of $10\,\Omega$ and an inductor of 0.5 H. If the initial current is 0, find the current at any time $t > 0$.

3. Answer any ***one*** question : 10×1=10

(a) Solve the differential equation of vibration of an infinite string $\frac{\partial^2 u}{\partial t^2} - C^2\frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty,$ $t > 0$.

P.T.O.

(b) Holling-Tanner type predator prey model is considered in the form

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{exy}{m+x},$$

$$\frac{dy}{dt} = Qy\left(1 - \frac{gy}{x}\right).$$

Describe the underlying assumption of this model. Find the interior equilibrium and give its stability conditions.