B.Sc./6th Sem (H)/MATH/23(CBCS)

2023

6th Semester Examination

MATHEMATICS (Honours)

Paper : DSE 3-T

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Mechanics]

Group - A

Answer any ***ten*** questions : $2\times10=20$

1. An artificial satellite revolves about the earth at a height H above the surface. Find the orbital speed so that a man in the satellite will be in a state of weightlessness.

2. Define 'apse' of a central orbit. Show that, at an apse, a particle is moving at right angles to the radius vector.

3. When the equilibrium of a rigid body under the action of a number of coplanar forces will be stable or unstable in nature?

P.T.O.

4. Find the C.G. of a uniform arc of a circle.

5. Show that the centres of suspension and oscillation of a compound pendulum are interchangeable.

6. Define equi-momental bodies. Write the conditions for equi-momental bodies.

7. Find moment of inertia of circular ring of mass M and of radius 'a' about any diameter.

8. Define compound pendulum. What do you mean by simple-equivalent pendulum?

9. State conservation of linear momentum under finite forces. Also state conservation of energy.

10. Prove that the momental ellipsoid at the centre of the elliptic plate whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z^2 = \text{constant}$.

11. A particle describes an ellipse about a focus and when at the end of minor axis receives a small impulse towards that focus which communicates a velocity u to the particle. Show that the eccentricity is increased by $ua\left(1-e^2\right)^{3/2}/h$.

12. A particle describes an ellipse under a force $\frac{\mu}{(\text{distance})^2}$ towards the focus; if it was projected with velocity v from a point at a distance r from the centre of force, show that its periodic time is $\frac{2\pi}{\mu}\left[\frac{2}{r}-\frac{v^2}{\mu}\right]^{-3/2}$.

13. Find the velocity of an artificial satellite of the earth, given $g = 9.8$ m/sec^2, radius of earth $= 6.4 \times 10^8$/metres. (Assume that the satellite is moving very close to the surface of the earth).

14. The position of a particle of mass m moving in space referred to a set of rectangular axes at any instant t is $\left(a\cos nt,\, a\sin nt,\, \frac{1}{2}at^2\right)$. Find the magnitude and direction of the acceleration.

15. What is meant by principal axes of a given material system at a point? State the condition so that a given straight line may be a principal axis of the material system at any point of its length.

P.T.O.

Group - B

Answer any ***four*** questions : $5\times4=20$

16. If a system of forces in one plane reduces to a couple whose moments is G and when each force is turned through a right angle it reduces to a couple H. Prove that when each force is turned through an angle α, the system is equivalent to a couple whose moment is $G\cos\alpha + H\sin\alpha$.

17. Find the C.G. of area enclosed by the curves $y^2 = ax$ and $x^2 + y^2 = 2ax$ lying in the first quadrant.

18. A particle is projected at right angle to the line joining it to a centre of force, attracting according to the law of inverse square of the distance, with a velocity $\frac{\sqrt{3}}{2}V$, where V denotes the velocity from infinity. Find the eccentricity of the orbit described and show that the periodic time is $2\pi T$, where T is the time taken to describe the major-axis of the orbit with velocity V.

19. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k-times the velocity. Show that the direction of the path described will again make an angle α with the horizon after a time $\frac{1}{k}\log\left(1+\frac{2ku}{g}\sin\alpha\right)$.

20. OA, OB, OC are the edges of a cube of side a and OO', AA', BB', CC' are its diagonals; along OB', $O'A$, BC, $C'A'$ act forces equal to P, $2P$, $3P$, $4P$; show that they are equivalent to force $\sqrt{35}P$ at O along a line whose direction cosines are proportional to -3, -5, 6 together with a couple $\frac{Pa}{2}\sqrt{114}$ about a line whose direction cosines are proportional to $7, -2, 2$.

21. Two equal uniform rods, AB and AC, are freely hinged at A and rest in a straight line on a smooth table. A blow is struck at B perpendicular to the rods, show that the kinetic energy generated is $\frac{7}{4}$ times what it would be if the rods were rigidly fastened together at A.

Group - C

Answer any ***two*** questions : 10×2=20

22. (i) A beam of length l rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizon are α and β, and the centre of gravity of the beam divides it in the ratio $a : b$. Find the position of equilibrium of the beam and show that the equilibrium is unstable. 6

P.T.O.

(ii) If a hemisphere rests in equilibrium with its curved surface in contact with a rough plane inclined to a horizontal at an angle θ then show that the inclination of the plane of the hemisphere to the horizontal is $\sin^{-1}\left(\frac{8}{3}\sin\theta\right)$, provided $\theta < \sin^{-1}\frac{3}{8}$. 4

23. (i) Prove that every given system of forces acting on a rigid body can be reduced to a wrench. 5

(ii) Six forces, each equal to P, act along the edges of a cube, taken in order which do not meet a given diagonal. Show that their resultant is a couple of moment $2\sqrt{3}Pa$, where a is the edge of the cube. 5

24. (i) Find the kinetic energy of a body moving in two dimensions. 4

(ii) A lamina in the form of an ellipse is rotating in its own plane about one of its foci with angular velocity ω. This focus is set free and the other, at the same instant is fixed, show that the ellipse now rotate about it with angular velocity $\omega\frac{2-5e^2}{2+3e^2}$. 6

25. (i) Having given the moments and products of inertia of a rigid body about three perpendicular concurrent axes. Find the moment of inertia of the body about an axis, with known direction cosines through that

point. Hence deduce the equation of momental ellipsoid of the body at that point. 3+2

(ii) A rough uniform rod of length $2a$ is placed on a rough table at right angles to its edge. If its C.G. be initially at a distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle $\tan^{-1}\left(\dfrac{\mu a^2}{a^2+9b^2}\right)$, where μ is the coefficient of friction. 5

P.T.O.

OR

[Number Theory]

Group - A

Answer any ***ten*** questions : 2×10=20

1. If an integer $a > 1$ is not divisible by any prime $p \leq \sqrt{a}$, then prove that a is a prime.

2. State the prime number theorem. What is interpretation of the theorem?

3. Show that the sum of twin primes p and $p+2$ is divisible by 12, provided that $p > 3$.

4. Prove that $ax \equiv ay \pmod{n} \Rightarrow x \equiv y \left(\bmod \dfrac{n}{\gcd(a,n)} \right)$.

5. Is $2(56!) + 1$ divisible by 59? Justify.

6. Prove that the product of the positive divisors of an integer $n > 1$ is equal to $n^{\frac{\tau(n)}{2}}$.

7. Find the number of zeros at the end of 1000!.

8. For any $n \in \mathbb{N}$, show that $\phi(n^2) = n\phi(n)$.

9. For any integer $n \geq 3$, show that $\sum_{k=1}^{n} \mu(k!) = 1$, where μ is the Möbius function.

10. Show that the Dirichlet inverse of λ is $|\mu|$, where λ and μ are Liouville function and Möbius function respectively.

11. If p is an odd prime, then prove that the only incongruent solutions of $x^2 \equiv 1 (\bmod p)$ are 1 and $p-1$.

12. Find all prime numbers that divide 50!.

13. Find the value of the following Legendre symbol : $\left(\frac{-23}{59}\right)$.

14. If p is an odd prime, then show that $\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)=0$.

15. Find last two digits of the number 9^{9^9}.

Group - B

Answer any *four* questions : 5×4=20

16. Show that a square-free number n is either prime or a Carmichael number if $p-1$ divides $n-1$ for every prime divisor p of n. Prove that a positive integer is divisible by 11 if and only if the sum of its digits with alternate signs in its decimal expansion is divisible by 11. 2+3

P.T.O.

17. Let F and f be two number theoretic functions related by the formula $F(n)=\Sigma_{d|n}f(d)$. Then show that

$$f(n)=\Sigma_{d|n}\mu(d)F\left(\frac{n}{d}\right)=\Sigma_{d|n}\mu\left(\frac{n}{d}\right)F(d).$$

18. How to encrypt and decrypt the message $M<n$ in the RSA cryptosystem with the are public key (n, e)? 5

19. Prove that an integer $n>1$ is prime if and only if $(n-2)!\equiv 1(\bmod n)$. 5

20. If p is a prime number and $d \mid p-1$, then show that the congruence $x^d-1\equiv 0(\bmod p)$ has exactly d solutions. 5

21. If g and h are multiplicative, then prove that the Dirichlet product $g*h$ is also multiplicative. Show that $\tau*\phi=\sigma$, where τ, ϕ, σ are divisor function, Euler's phi function, divisor sum function respectively. 3+2

Group - C

Answer any *two* questions : 10×2=20

22. Solve the system of linear congruences $2x\equiv 1(\bmod 5)$, $4x\equiv 1(\bmod 7)$, $5x\equiv 9(\bmod 11)$. For any positive integer n, show that $\Sigma_{d|n}\phi(d)=n$. State and prove Euler's theorem. 4+2+4

23. Verify that 2 is a primitive root of 19, but not of 17. If p is a prime and d is an integer dividing $p-1$, then prove that the polynomial congruence $x^d - 1 \equiv 0 \pmod{p}$ has exactly d solutions. Show that there is no primitive root modulo 2^e if $e \geq 3$. Solve the congruence $x^8 \equiv 5 \pmod{11}$. 2+3+3+2

24. If p be an odd prime and gcd $(a, p) = 1$, then prove that a is a quadratic residue or nonresidue of p according to whether $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ or $a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$. If p be an odd prime, then show that 2 is a quadratic residue modulo p if $p \equiv 1 \pmod{8}$ or $p \equiv 7 \pmod{8}$ and quadratic nonresidue modulo p if $p \equiv 5 \pmod{8}$ or $p \equiv 3 \pmod{8}$. 5+5

25. If p and q are distinct odd primes, then prove that

$$\left(\frac{p}{q}\right) = \begin{cases} \left(\frac{q}{p}\right) & \text{if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -\left(\frac{q}{p}\right) & \text{if } p \equiv q \equiv 3 \pmod{4} \end{cases}$$

If $p \neq 3$ is an odd prime, then show that

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{12} \\ -1 & \text{if } p \equiv \pm 5 \pmod{12} \end{cases}$$

P.T.O.

Is $x^2 \equiv 17\left(\bmod 2^5 \cdot 13^7 \cdot 47^{50}\right)$ is solvable? Justify. Show that $3n$, $4n$, $5n$ where $n = 1, 2, \ldots$ are the only Pythagorean triples whose terms are in arithmetic progression. 3+2+3+2

OR

[Industrial Mathematics]

Group - A

Answer any ***ten*** questions from the following :

$2 \times 10 = 20$

1. What mathematical technique is commonly used in medical imaging to reconstruct images from raw data obtained through techniques such as computed tomography (CT) or magnetic resonance imaging (MRI)?

2. What mathematical concept is utilized in medical image processing to enhance or filter images, remove noise, or improve image quality?

3. What mathematical method is often employed in medical image segmentation to identify and delineate regions of interest, such as tumors or blood vessels, from medical images?

4. What mathematical concept is utilized in X-ray imaging to create images of internal structures of objects or materials?

5. What mathematical method is commonly used in industrial X-ray imaging to analyze the defects or features of objects, such as welds, castings, or composites?

6. What mathematical tool is often employed in X-ray computed tomography (CT) for reconstructing cross-sectional images of objects from X-ray projections?

7. What is the Radon Transform?

8. What is the main application of the Radon Transform in industrial mathematics?

9. What are some properties of the Radon Transform that make it useful in industrial applications?

10. What is CT scan in the context of industrial mathematics?

11. What is Back Projection in the context of image processing and computed tomography (CT)?

12. What are some advantages of Back Projection in industrial applications?

13. What are some applications of CT scan in industrial settings?

14. What mathematical techniques are commonly used in CT scan image reconstruction in industrial mathematics?

15. What is an inverse problem in industrial mathematics?

Group - B

Answer any ***four*** questions from the following :

$5 \times 4 = 20$

16. How does Back Projection work in CT imaging?

17. What is the importance of inverse problem?

18. What are some common approaches for solving inverse problems in industrial mathematics?

19. What are some challenges associated with solving inverse problems in industrial mathematics?

20. Write short note on X-ray Behaviour.

21. Provide a framework of Inverse Problems.

Group - C

Answer any ***two*** questions from the following :

$10 \times 2 = 20$

22. Discuss Geological anomalies in Earth's interior from measurements at its surface.

23. Discuss Beers Law with a suitable illustration.

24. Mention various properties of Back Projection.

25. Mention the properties of Inverse Fourier Transforms in CT scan.
