B.Sc./6th Sem (H)/MATH/23(CBCS)

2023

6th Semester Examination

MATHEMATICS (Honours)

Paper : C 14-T

[Ring Theory and Linear Algebra-II]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any **ten** questions : 2×10=20

1. Let $Z_i = \{a+ib : a, b \in Z\}$ be the ring of Gaussian integers. Show that $1+i$ is irreducible element in Z_i.

2. Show that the polynomial $f(x) = x^2 + \overline{3}x + \overline{2} \in \mathbb{Z}_6[x]$ has four zeros in $\mathbb{Z}_6$.

3. Let W_1 and W_2 be subspaces of a finite dimensional inner product space. Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

4. Let T be a linear operator on a finite dimensional inner

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product space V. If T has an eigenvector, then its adjoint operator T^* does have so.

5. Let R be a UFD and $a, b, c \in R \setminus \{0\}$ such that $a|bc$ and $gcd(a, b) \sim 1$. Then prove that $a|c$.

6. State ***Eisenstein's criterion*** for irreducibility of a polynomial $f(x) \in \mathbb{Z}[x]$ over $\mathbb{Z}$.

7. Let U be a subset of a vector space V over the field F. Prove that the annihilator of U (denoted by U^0) is a subspace of the dual space V^*.

8. Show that the polynomial $f(x) = 21x^3 - 3x^2 + 2x + 9$ is irreducible over $\mathbb{Q}$.

9. For any linear transformation T, define the adjoint linear transformation T^* of T. Hence prove that

$$(T_1T_2)^* = T_2^*T_1^*$$

10. Let P be the linear operator on the vector space $\mathbb{R}^2$ over $\mathbb{R}$ defined by $P(x, y) = (x, 0)$ for all $(x, y) \in \mathbb{R}^2$. Find the minimal polynomial for P.

11. If T is a unitary linear transformation, then show that the characteristic roots of T all have absolute value 1.

12. Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$ equipped with

the standard inner product. Consider $\alpha = (1,2)$ and $\beta = (-1,1)$. Find an element $\gamma \in \mathbb{R}^2$ for which $<\alpha, \gamma> = -1$ and $<\beta, \gamma> = 3$.

13. Let V be an inner product space and $x, y \in V$. If $<x, v> = <y, v>$ for all $v \in V$ then prove that $x = y$.

14. Let V be the vector space $\mathbb{C}^2$ over $\mathbb{C}$ with the standard inner product. Let T be the linear operator on V defined by $T(1, 0) = (1, -2)$ and $T(0, 1) = (i, -1)$. Find $T^*(\alpha)$ where $\alpha = (x_1, x_2) \in V$.

15. Give an example of a 2×2 complex matrix A such that A^2 is normal but A is not normal.

Group - B

Answer any ***four*** questions : $5 \times 4 = 20$

16. (i) Let R be an integral domain and $f(x), g(x) \in R[x]$. Prove that

$$\deg(f(x)g(x)) = \deg f(x) + \deg g(x)$$

(ii) Find all the associates of $x^2 + [2]$ in $\mathbb{Z}_7[x]$.

3+2=5

17. (i) Is the ring of all 2×2 matrices with their entries from $\mathbb{Z}$ a PID? Justify your answer.

P.T.O.

(ii) Exhibit (with proper justification) an ideal I in the polynomial ring $\mathbb{Z}[x]$ so that I is not a principal ideal. 3+2=5

18. Let V be the vector space of all polynomials over $\mathbb{R}$ with degree less than or equal to 2. Let t_1, t_2, t_3 be three distinct real numbers and L_1, L_2, L_3 be three linear functionals on V defined by $L_i(p) = p(t_i)$ for all $i = 1,2,3$. Find the basis $\mathcal{B} = \{p_1, p_2, p_3\}$ of V such that $\{L_1, L_2, L_3\}$ becomes the dual basis of $\mathcal{B}^*$ of $\mathcal{B}$.

19. Consider the matrix $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$ over the field of real numbers. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

20. Let V be a finite dimensional vector space over the field F and T be a linear operator on V. If T is diagonalizable, then prove that the minimal polynomial for T is a product of distinct linear factors.

21. Let V be a finite dimensional inner product space and T be an invertible linear operator on V. Then show that

(i) T^* is also an invertible operator on V.

(ii) $(T^*)^{-1} = (T^{-1})^*$. 2+3=5

Group - C

Answer any *two* questions : 10×2=20

22. (i) Check whether the following statement is true or false : "Let R be an integral domain, $a, b \in R \setminus \{0\}$ and $d = \gcd(a, b)$. Then there exist $x, y \in R$ such that $d = ax + by$." Give proper justification in support of your answer. 4

(ii) Let V be a vector space of dimension m over a field F and W be a vector subspace of dimension k of V where $1 \leq k < n$. Then prove that the dimension of the subspace $\{f \mid f \in V^*, f(w) = 0 \forall w \in W\}$ is $n - k$. 4

(iii) Let V be a complex or real inner product space. Then show that the induced norm $\| \ \|$ satisfies the following equality :

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + 2\|y\|^2\right)$$

(***Parallelogram Law***) for all $x, y \in V$. 2

23. (i) Construct a field of 8 elements. 5

(ii) Let V be the vector space of all polynomial functions R to R of degree ≤ 2. Let t_1, t_2, t_3 be three distinct real numbers and let $L_i : V \to F$ be

P.T.O.

such that $L_i\left(p(x)\right)=p(t_i)$, $i=1,2,3$. Show that $\{L_1, L_2, L_3\}$ is a basis of $\widehat{V}$. Determine a basis of V such that $\{L_1, L_2, L_3\}$ is its dual. 5

24. (i) Prove that $I=<x^2+1>$ is a prime ideal in $\mathbb{Z}[x]$ but not a maximal ideal in $\mathbb{Z}[x]$. 5

(ii) Let W be the plane in $\mathbb{R}^3$ spanned by the set $S=\{(1,2,2),(-1,0,2)\}$. Then find an orthonormal basis $\mathcal{B}'$ for W applying Gram-Schmidt orthogonalization process and extend $\mathcal{B}'$ to an orthonormal basis $\mathcal{B}$ of V. 3+2=5

25. (i) For any prime p, show that the p^{th} cyclotomic polynomial is irreducible over $\mathbb{Q}$.

(ii) Let V be an inner product space and $S=\{v_1, v_2, \ldots\ldots, v_n\}$ be an orthonormal subset of V. Show that for any $x\in V$, $\|x\|^2 \geq \sum_{i=1}^{n} |<x, v_i>|^2$. 4+6