

2023

6th Semester Examination
MATHEMATICS (Honours)

Paper : C 13-T

[Metric Spaces and Complex Analysis]

[CBCS]

Full Marks : 60

Time : Three Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

Group - A

Answer any **ten** questions : $2 \times 10 = 20$

1. State the Banach fixed point theorem.
2. What do you mean by complete metric space?
3. Define uniform continuity.
4. Write down the Heine-Borel property.
5. Let (X, d) be a metric space in which A and B are two intersecting connected sets. Show that $A \cup B$ is connected.

P.T.O.

$$6. \text{ Let } f(z) = \frac{|z|}{\operatorname{Re}(z)} \quad \text{if } \operatorname{Re}(z) \neq 0$$

$$= 0 \quad \text{if } \operatorname{Re}(z) = 0$$

Show that $f(z)$ is not continuous at $z = 0$.

7. Show that the function $u = \cos x \cosh y$ is harmonic.

8. Find the radius of convergence $\sum_{n=2}^{\infty} \frac{z^n}{n(\log n)^2}$.

9. Evaluate $\oint_c \frac{1}{z} dz$, $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

10. Show that $u(x, y) = 4xy - x^3 + 3xy^2$, is harmonic.

11. Show that a convergent sequence in a metric space is bounded.

12. Give an example, in the real time $\mathbb{R}$, of the sequence $\{x_n\}$ such that $|x_n - x_{n+1}| \rightarrow 0$ (as $n \rightarrow \infty$) but $\{x_n\}$ is not Cauchy.

13. Show that for any subset A of a metric space (X, d) , the function $f: X \rightarrow \mathbb{R}$ given by $f(x) = d(x, A)$, $x \in X$, is uniformly continuous.

14. Show that $\lim_{z \rightarrow z_0} f(z)g(z) = 0$ if $\lim_{z \rightarrow z_0} f(z) = 0$ and if there exists a positive integer M such that $|g(z)| \leq M$ for all z in some neighbourhood of z_0 .

(3)

15. Let $T(z) = \frac{az+b}{cz+d}$ be a bilinear transformation. Show that ∞ is a fixed point of T if and only if $c = 0$.

Group - B

Answer any *four* questions : $5 \times 4 = 20$

16. Show that continuous image of a compact metric space is compact.
17. Check whether the function is differentiable at $z = 0$. Also check whether it satisfies C-R equations.

$$\begin{aligned} f(z) &= \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}} \quad (z \neq 0) \\ &= 0 \quad (z = 0) \end{aligned}$$

18. If $f(z)$ is an alytic function within and on a closed contour C , and if a is any point within C , then show that

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

19. (i) Prove that a metric space (X, d) having the property that every continuous map $f: X \rightarrow X$ has a fixed point, is connected. 2
- (ii) Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a contraction on X . Then for $x \in X$, show that the sequence $\{T^n(x)\}$ is a convergent sequence. 3

P.T.O.

20. (i) Determine whether the set $S = \{(x, y): 0 < x \leq 1, x^2 + y^2 = 4\}$ is compact in $\mathbb{R}^2$. 3

(ii) Let X be an infinite set endowed with the discrete metric. Show that every infinite subset of (X, d) is bounded but not totally bounded. 2

21. Evaluate :

(i) $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$ described in the positive sense. 2

(ii) $\int_C \frac{z dz}{(9-z^2)(z+i)}$ where C is the circle $|z| = 2$ described in the positive sense. 3

Group - C

Answer any **two** questions : $10 \times 2 = 20$

22. (i) State and prove Liouville's theorem.

(ii) If $f(z)$ is a regular analytic function of z , prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2.$$

23. (i) Show that any compact subset of a metric space is closed and bounded.

(ii) Show that two metrics d_1, d_2 on a set X are equivalent iff the identity map $I_X: (X, d_1) \rightarrow (X, d_2)$ is a homomorphism.

24. (i) Show that the map $f:[0,1] \rightarrow [0,1]$, defined by

$$f(x) = x - \frac{x^2}{2}, \quad x \in [0,1] \text{ is a weak contraction}$$

but not a contraction map. 3

- (ii) Let (X, d) be a complete metric space and $f:X \rightarrow X$ be a contraction map with Lipschitz constant t ($0 < t < 1$). If $x_0 \in X$ is the unique fixed

$$\text{point of } f, \text{ show that } d(x, x_0) \leq \frac{1}{1-t} d(x, f(x)),$$

for all $x \in X$. 5

- (iii) Show that a contraction of a bounded plane set may have the same diameter as the set itself. 2

25. (i) Let $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$ and $z_0 = x_0 + iy_0$. Let the function f be defined in a domain D except possibly at the point z_0 in D . Then prove that $\lim_{z \rightarrow z_0} f(z) = u_0 + iv_0$ if and only if

$$\lim_{x \rightarrow x_0} u(x, y) = u_0 \text{ and } \lim_{y \rightarrow y_0} v(x, y) = v_0. \quad 5$$

- (ii) Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}. \quad 5$$