

2022

5th Semester Examination
MATHEMATICS (General)

Paper : DSE 1A/2A/3A-T

[CBCS]

Full Marks : 60

Time : Three Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

[Complex Analysis]

1. Answer any **ten** questions : 2×10=20

(a) Show that $f(z) = |z|^2$ is continuous for all $z \in \mathbb{C}$.

(b) Define analytic function in complex plane. Give an example.

(c) If $z_1 = -1 + i$ and $z_2 = 2 - 3i$, evaluate $\operatorname{Im} \left(\frac{\bar{z}_1}{iz_2} \right)$.

(d) Prove that $f(z) = \bar{z}$ is nowhere differentiable on $\mathbb{C}$.

(e) State Cauchy-Riemann equation.

(f) Define harmonic function.

(g) Show that $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$ does not exist.

P.T.O.

(h) Define extended complex plane.

(i) Show that if $\lim_{z \rightarrow z_0} f(z)$ exists, then it must be unique.

(j) Prove that the function

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

satisfies Laplace's equation.

(k) State the Liouville's theorem.

(l) Give examples of absolute and uniform convergence of power series.

(m) Show that the function $f(z) = \frac{z - \sin z}{z^3}$ has a removable singularity at $z = 0$.

(n) Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series about $z = 0$.

(o) Find the domain of convergence of power series

$$\sum \left(1 - \frac{1}{n}\right)^{n^2} \cdot z^n.$$

2. Answer any **four** questions :

5×4=20

(a) If the real part of the complex no $\frac{z-i}{z-1}$ is zero,

then show that the complex no z lies on the circle

with centre $\frac{1+i}{2}$ and radius $\frac{1}{\sqrt{2}}$.

(b) Consider the function f defined by

$$f(z) = \begin{cases} 0, & \text{when } z = 0 \\ \frac{x^3 - y^3}{x^2 + y^2} + i \frac{(x^3 + y^3)}{x^2 + y^2}, & \text{when } z \neq 0 \end{cases}$$

Show that the function f satisfies the Cauchy-Riemann equation at the origin, but f is not differentiable at $z = 0$.

(c) Show that an analytic function with constant modulus is constant.

(d) Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \equiv 4 \frac{\partial^2}{\partial z \partial \bar{z}}$.

(e) Construct an analytic function $f(z)$ whose real part is $e^x \cos y$.

(f) State and prove the fundamental theorem of integral calculus in the complex plane.

3. Answer any **two** questions :

10×2=20

(a) (i) Find the residue of $F(z) = \frac{\cot z \cdot \coth z}{z^3}$ at $z = 0$.

P.T.O.

(ii) Evaluate $\int_0^{\infty} \frac{dx}{x^6 + 1}$.

(b) (i) State and prove Laurent's theorem.

(ii) Prove that an absolute convergent series is convergent.

(c) (i) Let $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$ and $z_0 = x_0 + iy$. Let the function f be defined in a domain except possibly at the point z_0 in D .

Then $\lim_{z \rightarrow z_0} f(z) = w_0 = u_0 + iv_0$ iff

$$\lim_{z \rightarrow z_0} u(x, y) = u_0 \text{ and } \lim_{z \rightarrow z_0} v(x, y) = v_0.$$

(ii) Let $f(z) = \begin{cases} \frac{|z|}{\operatorname{Re}(z)} & \text{if } \operatorname{Re}(z) \neq 0 \\ 0 & \text{if } \operatorname{Re}(z) = 0. \end{cases}$

Show that f is not continuous at $z = 0$.

(d) (i) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series for $|z| < 1$.

(ii) Show that the function $f(z) = \sqrt{xy}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied at the point.

OR

[Matrices]1. Answer any *ten* questions : $2 \times 10 = 20$

(a) For what value of k , the following system of equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$ possesses a non-trivial solution.

(b) Let V be a real vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Prove that the set $\{\alpha + \beta + \gamma, \beta + \gamma, \gamma\}$ is also a basis.

(c) Let $S = \{(x, y, z) \in R^3 : x + y + z = 0\}$, prove that S is a subspace of R^3 .

(d) Prove that an elementary row operation of the first kind does not alter the row rank of a matrix.

(e) Prove that a matrix and its transpose have the same eigenvalues.

(f) Determine the value for which the system of equations

$$x + y + z = 6$$

$$2x + y + 3z = 13$$

$$5x + 2y + az = 33,$$

has only one solution.

P.T.O.

(g) Find the eigen values of the matrix $A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$.

(h) Find the all real λ for which the rank of the following matrix A is 2 :

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \lambda \\ 1 & 1 & 6 & \lambda+1 \end{bmatrix}$$

(i) Verify Cayley-Hamilton theorem for the matrix :

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

(j) If A is an $n \times n$ matrix and there exists a unique matrix B such that $AB = I_n$, then prove that $BA = I_n$.

(k) In R^2 , if $\alpha = (3, 1)$, $\beta = (2, -1)$, then determine $L\{\alpha, \beta\}$.

(l) Show that the matrix $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is not diagonalizable.

(m) Express $(5, 2, 1)$ as a linear combination of $(1, 4, 0)$, $(2, 2, 1)$ and $(3, 0, 1)$.

(n) Find a basis and the dimension of the subspace W of R^3 , where $W = \{(x, y, z) \in R^3; x + y + z = 0\}$.

- (o) Let P is a real orthogonal matrix with $\det P = -1$.
Prove that -1 is an eigen value of P .

2. Answer any **four** questions :

5×4=20

(a) If $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$,

show that $A^2 - 10A + 16I_3 = 0$ and then find A^{-1} .

2+3

- (b) Prove that a set of vectors containing null vector is linearly dependent.

5

- (c) Find all real values of α for which the rank of the

matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \alpha \\ 5 & 7 & 1 & \alpha^2 \end{bmatrix}$ is 2.

5

(d) If $A = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$ is a 3×3 real matrix, then

show that 0 is an eigen value of the matrix A and also find the eigen vector corresponding to the eigen value 0.

2+3

P.T.O.

(e) Diagonalise the symmetric matrix $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.

5

(f) Prove that the row rank and column rank of any matrix are identical.

5

3. Answer any *two* questions : $10 \times 2 = 20$

(a) (i) Define the eigenbasis for a square matrix. 2

(ii) For the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find

eigenbasis in R^3 , if exists. 8

(b) (i) Determine the conditions for which the system $x + y + z = 1$, $x + 2y - z = b$, $5x + 7y + az = b^2$ admits of (i) only one solution, (ii) no solution, (iii) many solutions.

(ii) If λ be an eigen value of a non-singular matrix A , then prove that λ^{-1} is an eigen value of A^{-1} . 7+3

(c) (i) Define diagonalizable matrix. 2

(ii) If possible diagonalize the matrix

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$
. Also find the diagonalizing matrix, if exists. 8

- (d) (i) Find the inverse of the matrix A by elementary row operations, where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}$$

- (ii) Prove that the set $S = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$ is a basis of R^3 . 6+4

P.T.O.

OR

[Linear Algebra]1. Answer any *ten* questions :

2×10=20

- (a) For a vector space over the field F , prove that $cv = \theta \Rightarrow$ either $c = 0$ or $v = \theta$.
- (b) Define subspace of a vector space, V over the field F .
- (c) If S and T are two non-empty finite subsets of a vector space $V(F)$ and $S \subset T$, then prove that $L(S) \subset L(T)$.
- (d) Examine that the subset $\{(1, 2, 0), (3, -1, 1), (4, 1, 1)\}$ is linearly dependent or not.
- (e) Prove that union of two subspaces of a vector space $V(F)$ may not be a subspace of $V(F)$.
- (f) Find the basis of the subspace W of R^3 , when $W = \{(x, y, z) | x + 2y - 3z = 0\}$.
- (g) Let $V(F)$ be a vector space of dimension n . Prove that any linearly independent set of n vectors of V is a basis of V .
- (h) Define quotient subspace.
- (i) Define linear transformation from a vector space V to W over the field F .

- (j) Prove that a linear transformation $T:V \rightarrow W$ is injective iff $\ker(T) = \{\theta\}$.
- (k) Prove that $\dim V$ is an odd integer if $\dim \ker T = \dim \operatorname{Im} T$ for a linear transformation $T:V \rightarrow W$.
- (l) Find the matrix of the linear transformation $T:R^3 \rightarrow R^2$ is defined by $T(x,y,z) = (3x-2y+z, x-3y-2z)$, $(x,y,z) \in R^3$ with relative to the order basis $\{(0, 1, 0), (1, 0, 0), (0, 0, 1)\}$ of R^3 and $\{(0, 1), (1, 0)\}$.
- (m) Let V and W be the vector space over the field of F . If a linear mapping $T:V \rightarrow W$ be invertible then prove that the inverse mapping $T^{-1}:W \rightarrow V$ is linear.
- (n) A mapping $T:R^3 \rightarrow R^3$ is defined by $T(x,y,z) = (yz, zx, xy)$ where $(x,y,z) \in R^3$. Examine whether T is linear or not.
- (o) Prove that a linear mapping $f:R^2 \rightarrow R^2$ defined by $f(x,y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha)$ is an isomorphism. (α is a given constant)

2. Answer any **four** questions :

5×4=20

- (a) If $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of a vector space $V(F)$ and a non-zero vector ξ of V is expressed as $\xi = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$, $c_i \in F$, then

P.T.O.

$c_k \neq 0$ prove that $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \xi, \alpha_{k+1}, \dots, \alpha_n\}$ is a new basis of V .

(b) Let $V(F)$ be a vector space of a finite dimension and W is a subspace of V . Then prove that $\dim W \leq \dim V$.

(c) Let W be a subspace of a vector space $V(F)$. Prove that for $\alpha, \beta \in V$ the cosets $\alpha + W$ and $\beta + W$ are equivalent iff $\alpha - \beta \in W$.

(d) Find the linear transformation $T: R^3 \rightarrow R^2$ which maps the basis vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of R^3 to the vectors $(1, 1)$, $(2, 3)$ and $(3, 2)$ of R^2 respectively.

(e) Let V be a finite dimensional vector space over the field F and $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be an ordered basis of V . A linear mapping $T: V \rightarrow V$ as such that $T(\alpha_i) = \alpha_{i+1}$ for $i = 1, 2, \dots, n-1$ and $T(\alpha_n) = \alpha_1$. Prove that $T^n = I$, I being the identity mapping on V .

(f) Find the linear mapping $T: R^3 \rightarrow R^3$ if $T(1, 0, 0) = (2, 3, 4)$, $T(0, 1, 0) = (1, 2, 3)$, $T(0, 0, 1) = (1, 1, 1)$. Find the matrix T relative to the ordered basis $(\epsilon_1, \epsilon_2, \epsilon_3)$ where $\epsilon_1 = (1, 0, 0)$, $\epsilon_2 = (0, 1, 0)$, $\epsilon_3 = (0, 0, 1)$. Deduce that T is not invertible.

3. Answer any *two* questions : 10×2=20

(a) (i) State and prove the necessary and sufficient condition for a nonempty subset of a vector space $V(F)$ is subspace of $V(F)$. 4

(ii) Let $W_1 = L \{(1, -2, -1), (2, 3, 5)\}$ and $W_2 = L \{(1, -2, 0), (3, -3, 0)\}$, then show that W_1 and W_2 are two subspaces of $V_3(R)$. Determine $\dim W_1$, $\dim W_2$, $\dim W_1 \cap W_2$ and hence deduce that $\dim(W_1 + W_2) = 3$. 2+4

(b) (i) Let V is a finite dimensional vector space over the field F . Define null space and rank (T) for a linear transformation $T: V(F) \rightarrow W(F)$. Prove that $\text{rank}(T) + \text{nullity}(T) = \dim V$. 2+5

(ii) Find a basis for the null space of the matrix

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}. \quad 3$$

(c) (i) Prove that $\ker T$ of a linear mapping $T: V \rightarrow W$ is subspace of V . 3

(ii) Find the linear mapping $T: R^3 \rightarrow R^3$ such that $\text{Im}(T)$ is the subspace $W = \{(x_1, x_2, x_3) \in R^3 : x_1 + x_2 - x_3 = 0\}$ in R^3 . 3

P.T.O.

(iii) Let V be a vector space of all $n \times n$ matrices over the field F , and let B be a fixed $n \times n$ matrix. If $T(A) = AB - BA$ verify that T is a linear transformation from V into V . 4

(d) (i) Define isomorphism for a linear transformation $T: V \rightarrow W$. Let V and W be finite dimensional vector spaces over the field F . Prove that V and W are isomorphic iff $\dim V = \dim W$. 2+4

(ii) Let (x, y, z) be an ordered basis of a real vector space $V(F)$ and a linear mapping $T: V \rightarrow V$ is defined by $T(x) = x + y + z$, $T(y) = x + y$, $T(z) = x$. Find the matrix of T^{-1} . 4

OR

[Vector Calculus and Analytical Geometry]1. Answer any *ten* from the following questions : $2 \times 10 = 20$

(a) Determine the points of intersection of the line

$$\frac{x+2}{-1} = \frac{y+4}{-2} = \frac{z-3}{1} \text{ and the cylinder } x^2 + z^2 = 1.$$

(b) Find the length of the chord intercepted by the parabola $y^2 = 4ax$ on the straight line $y = 3x + 2$.(c) Find the eccentric angle of a point on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the center.(d) Prove that the parametric equations $x = \frac{a}{2} \left(\lambda + \frac{1}{\lambda} \right)$

$$\text{and } y = \frac{b}{2} \left(\lambda - \frac{1}{\lambda} \right) \text{ represent a hyperbola.}$$

(e) What type of conic does $x^2 + 2y^2 = 3$ represent?(f) Show that $[\alpha + \beta, \beta + \gamma, \gamma + \alpha] = 2[\alpha, \beta, \gamma]$, where α, β, γ are any three vectors.(g) Show that the vectors $A = 2i - j + k$, $B = i - 3j - 5k$, $C = 3i - 4j - 4k$, form the sides of a right-angled triangle.

P.T.O.

(h) If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors, no two of them are collinear and the vector $\vec{a} + \vec{b}$ is collinear with $\vec{c}$; $\vec{b} + \vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

(i) If PSP' be the focal chord of a conic

$$\frac{l}{r} = 1 - e \cos \theta, \text{ then show that } \frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}.$$

(j) If $\vec{a} = (1, 0, 5)$. Find the unit vector of $\vec{a}$.

(k) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$, then prove that $\vec{\nabla}(f(r)) = f'(r)\vec{\nabla}r$.

(l) Find the equation of the sphere described on the join of $P(2, -3, 4)$ and $Q(-1, 0, 5)$ as diameter.

(m) Prove that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{f}) = 0$.

(n) The eccentricities of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ are e and e' respectively. Prove

$$\text{that } \frac{1}{e^2} - \frac{1}{(e')^2} = 1.$$

- (o) $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 6\hat{i} + 9\hat{j} - 3\hat{k}$. Then find $\vec{a} \times \vec{b}$.

2. Answer any **four** from the following questions : $5 \times 4 = 20$

- (a) Show that the locus of the pole w.r.t. the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ of any tangent to the director circle}$$

$$\text{of the ellipse } \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1 \text{ is } \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2 + d^2}.$$

- (b) Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k}$ is a conservative force field. Find the scalar potential for $\vec{F}$. Also, find the work done in moving a particle in this field from $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$.

- (c) Prove that an angle inscribed in a semi-circle is a right angle.

- (d) Show that the vector $\frac{1}{r^3} \vec{r}$ is solenoidal where

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \text{ and } |\vec{r}| = r.$$

- (e) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle.

P.T.O.

- (f) Determine the locus of the points of intersection of perpendicular generators of the paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z.$$

3. Answer any *two* questions :

10×2=20

- (a) (i) If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then prove that the vectors

$\vec{a}$ and $\vec{b}$ are orthogonal. 5

- (ii) If $\vec{r} = (a \cos t)\vec{i} + (a \sin t)\vec{j} + (at \tan \alpha)\vec{k}$,

then show that $\left| \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right| = a^2 \sec \alpha$. 5

- (b) (i) Establish the following relation :

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\vec{\nabla}^2 \vec{A} + \vec{\nabla} (\vec{\nabla} \cdot \vec{A}). \quad 5$$

- (ii) If θ is the angle between the surfaces

$$xy^2z = 3x + z^2 \quad \text{and} \quad 3x^2 - y^2 + 2z = 1 \quad \text{at the}$$

point $(1, -2, 1)$, show that $\cos \theta = \frac{3}{7\sqrt{6}}$. 5

- (c) (i) Tangent are drawn to the parabola $y^2 = 4ax$ at the points whose abscissa are in the ratio $p : 1$. Show that the locus of their point of intersection is a parabola. 8

(ii) Find a generator of the cone $5yz^2 + x^2 = 0$.
2

(d) A sphere of constant radius r passes through the origin and cuts the axes in A, B, C . Prove that the locus of the foot of the perpendicular from origin to the plane ABC is given by

$$(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = 4r^2. \quad 10$$
